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Hodograph transformation methods in non-Newtonian fluids

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Summary

Solutions for the equations of motion of an incompressible second-grade fluid are obtained by employing hodograph-transformation methods. By introducing a suitable Legendre-transform function the basic equations are recast in terms of this function, and the conditions which this function should satisfy are stated. Several illustrations of the method are considered and the results for stream-lines, velocities and pressure distribution are compared with the corresponding results for viscous fluids.

1. Introduction

In recent years, transformation techniques have become some of the powerful methods for solving non-linear partial differential equations. Amongst many, the hodograph transformations have gained considerable success in gas-dynamics problems. Ames [1] has given an excellent survey of this method together with applications in various other fields. Recently, Chandna, Barron and Smith [2] have used the hodograph and Legendre transformations to study plane steady viscous flow problems.

In the present paper we also employ hodograph and Legendre transformations to study the flow problems in a second-grade fluid [3]. We first consider the interchange of dependent and independent variables and then introduce a Legendre-transform function of the stream function and recast all the equations in terms of this transformed function. The equation this function must satisfy is then determined and several illustrations to display the use of the method are considered. With regard to the streamlines and velocity distribution we find that some of the results for the viscous fluids hold also for second-grade fluids. In some cases we find that the non-Newtonian nature of the fluid eliminates certain flows which are otherwise possible in Newtonian fluids. The dynamic pressure distribution, in almost all cases, appears to be different to that obtained for viscous fluids.

We point out that our approach is an inverse method in the sense that we select a form for the Legendre-transform function and then find conditions when such a function will be possible for physically meaningful situations. We then determine the stream function, velocity components and pressure distribution, via certain suitable relations, for such possible cases. The importance of the inverse methods in non-Newtonian fluids has recently been pointed out in the related work [4].

2. Basic equations

The basic equations governing the motion of a homogeneous incompressible second-grade fluid, neglecting thermal effects, are

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1}$$

$$\operatorname{div} \boldsymbol{T} + \rho \boldsymbol{f} = \rho \dot{\boldsymbol{v}}, \tag{2}$$

and the constitutive equation for the Cauchy stress $\boldsymbol{T}$ [3],

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu \boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2. \tag{3}$$

Here $\boldsymbol{v}$ is the velocity, ρ the density, $\boldsymbol{f}$ the body force per unit mass, p the dynamic pressure, μ the coefficient of dynamic viscosity and α_1 and α_2 are the normal-stress moduli. The Rivlin-Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined as

$$\boldsymbol{A}_1 = (\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^{\mathrm{T}}, \qquad \boldsymbol{A}_2 = \dot{\boldsymbol{A}}_1 + (\nabla \boldsymbol{v})^{\mathrm{T}} \boldsymbol{A}_1 + \boldsymbol{A}_1 (\nabla \boldsymbol{v}). \tag{4}$$

If we substitute (3) in (2) and make use of (4) we get

$$-\operatorname{grad} p + \mu \nabla^2 \boldsymbol{v} + \alpha_1 \left[\nabla^2 \boldsymbol{v}_t + \nabla^2 (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} \right.$$
$$\left. + \operatorname{grad}\left(\boldsymbol{v} \cdot \nabla^2 \boldsymbol{v} + \tfrac{1}{4} |\boldsymbol{A}_1|^2 \right) \right] + (\alpha_1 + \alpha_2) \operatorname{div} \boldsymbol{A}_1^2 + \rho \boldsymbol{f} = \rho \dot{\boldsymbol{v}}, \tag{5}$$

where ∇^2 denotes the Laplacian, $\boldsymbol{v}_t$ denotes the partial derivative of $\boldsymbol{v}$ with respect to time and

$$|\boldsymbol{A}_1|^2 = \operatorname{tr} \boldsymbol{A}_1 \boldsymbol{A}_1^{\mathrm{T}}.$$

In the case of steady plane flow, when body forces are absent, (1) and (5) reduce to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6}$$

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] + \frac{\partial p}{\partial x}$$
$$= \mu \nabla^2 u + \alpha_1 \left[\frac{\partial}{\partial x} \left\{ 2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left(\frac{\partial u}{\partial x} \right)^2 \right. \right.$$
$$\left. + 2 \frac{\partial v}{\partial x} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right\} + \frac{\partial}{\partial y} \left\{ \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right.$$
$$\left. \left. + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right\} \right] + \alpha_2 \left[\frac{\partial}{\partial x} \left\{ 4 \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right\} \right], \tag{7}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{\partial p}{\partial y}$$

$$=\mu\nabla^2 v+\alpha_1\left[\frac{\partial}{\partial x}\left\{\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right.\right.$$

$$\left.+2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right\}+\frac{\partial}{\partial y}\left\{2u\frac{\partial^2 v}{\partial x\partial y}+2v\frac{\partial^2 v}{\partial y^2}+4\left(\frac{\partial v}{\partial y}\right)^2\right.$$

$$\left.\left.+2\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}\right]+\alpha_2\left[\frac{\partial}{\partial y}\left\{4\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\}\right]. \tag{8}$$

Equations (6)–(8) are three partial differential equations for three unknowns: u, v and p. We introduce the two-dimensional vorticity function ω and a generalised energy function h as

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{9}$$

$$h=\tfrac{1}{2}\rho q^2-\alpha_1(u\nabla^2 u+v\nabla^2 v)-\tfrac{1}{4}(3\alpha_1+2\alpha_2)|A_1|^2+p, \tag{10}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\qquad q^2=u^2+v^2$$

and

$$|A_1|^2=4\left(\frac{\partial u}{\partial x}\right)^2+4\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2.$$

When (9) and (10) are employed in (7) and (8) we find that (6)–(8) are replaced by a system of four partial differential equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega.$$

$$\begin{aligned}\frac{\partial h}{\partial x}&=\rho v\omega-\mu\frac{\partial\omega}{\partial y}-\alpha_1 v\nabla^2\omega,\\ \frac{\partial h}{\partial y}&=-\rho u\omega+\mu\frac{\partial\omega}{\partial x}+\alpha_1 u\nabla^2\omega,\end{aligned} \tag{11}$$

for the four unknown functions u, v, ω, h, of (x, y). Once a solution for these is determined, the pressure p is obtained from the generalised energy expression (10).

3. Equations in the hodograph plane

Let the flow variables $u(x, y)$, $v(x, y)$ be such that, in the flow region under consideration, the Jacobian

$$J=\frac{\partial(u, v)}{\partial(x, y)} \neq 0, \tag{12}$$

satisfies $0<|J|<\infty$. In such cases we may consider x and y as functions of u and v and the following relations hold:

$$\begin{aligned} &\frac{\partial u}{\partial x}=J\frac{\partial y}{\partial v}, \qquad \frac{\partial u}{\partial y}=-J\frac{\partial x}{\partial v},\\ &\frac{\partial v}{\partial x}=-J\frac{\partial y}{\partial u}, \qquad \frac{\partial v}{\partial y}=J\frac{\partial x}{\partial u}, \end{aligned} \tag{13}$$

$$J(x, y)=\frac{\partial(u, v)}{\partial(x, y)}=\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^{-1}=j(u, v), \tag{14}$$

$$\begin{aligned} &\frac{\partial f}{\partial x}=\frac{\partial(f, y)}{\partial(x, y)}=J\frac{\partial(f, y)}{\partial(u, v)}=j\frac{\partial(f, y)}{\partial(u, v)},\\ &\frac{\partial f}{\partial y}=-\frac{\partial(f, x)}{\partial(x, y)}=J\frac{\partial(x, f)}{\partial(u, v)}=j\frac{\partial(x, f)}{\partial(u, v)}, \end{aligned} \tag{15}$$

where $f=f(x, y)$ is any continuously differentiable function and $f(u, v)$ is its transformed function in the (u, v) plane.

Now we take up the four equations (11) and employ the above transformations in these equations. We find that the transformed system of equations in the hodograph plane (u, v) is given as

$$\frac{\partial y}{\partial v}+\frac{\partial x}{\partial u}=0, \tag{16}$$

$$j\left(\frac{\partial x}{\partial v}-\frac{\partial y}{\partial u}\right)=\omega, \tag{17}$$

$$-j\frac{\partial(h, y)}{\partial(u, v)}=-\rho v\omega+\mu jW_1+\alpha_1 vj\left[\frac{\partial(x, jW_1)}{\partial(u, v)}+\frac{\partial(-y, jW_2)}{\partial(u, v)}\right], \tag{18}$$

$$j\frac{\partial(x, h)}{\partial(u, v)}=-\rho u\omega+\mu jW_2+\alpha_1 uj\left[\frac{\partial(x, jW_1)}{\partial(u, v)}+\frac{\partial(-y, jW_2)}{\partial(u, v)}\right], \tag{19}$$

where

$$W_1=W_1(u, v)=\frac{\partial(x, \omega)}{\partial(u, v)}, \qquad W_2=W_2(u, v)=\frac{\partial(-y, \omega)}{\partial(u, v)}. \tag{20}$$

This is a system of four partial differential equations in the four unknown functions x, y, ω, h, of (u, v). Once a solution $x = x(u, v)$, $y = y(u, v)$, $\omega = \omega(u, v)$, $h = h(u, v)$ is determined, we are lead to the solutions $u = u(x, y)$, $v = v(x, y)$ and therefore $\omega = \omega(x, y)$, $h = h(x, y)$ for the system (11). Equation (16) implies the existence of a function $L(u, v)$, called the Legendre-transform function of the stream function $\psi(x, y)$, such that

$$L_u = \frac{\partial L}{\partial u} = -y, \qquad L_v = \frac{\partial L}{\partial v} = x \tag{21}$$

and

$$L(u, v) = vx - uy + \psi(x, y). \tag{22}$$

On introducing $L(u, v)$ as defined by (21) or (22), we eliminate $x(u, v)$ and $y(u, v)$ from (16)–(19). We find that (16) is identically satisfied and the other equations take the form

$$j[L_{uu} + L_{vv}] = \omega(u, v). \tag{23}$$

$$j\frac{\partial(h, L_u)}{\partial(u, v)} = -\rho v\omega + \mu j W_1 + \alpha_1 vj\left[\frac{\partial(L_v, jW_1)}{\partial(u, v)} + \frac{\partial(L_u, jW_2)}{\partial(u, v)}\right], \tag{24}$$

$$j\frac{\partial(L_v, h)}{\partial(u, v)} = -\rho u\omega + \mu j W_2 + \alpha_1 uj\left[\frac{\partial(L_v, jW_1)}{\partial(u, v)} + \frac{\partial(L_u, jW_2)}{\partial(u, v)}\right], \tag{25}$$

where now

$$W_1 = \frac{\partial(L_v, \omega)}{\partial(u, v)}, \qquad W_2 = \frac{\partial(L_u, \omega)}{\partial(u, v)} \tag{26}$$

and

$$j = \left[L_{uu}L_{vv} - L_{uv}^2\right]^{-1}. \tag{27}$$

Now we make use of the integrability condition

$$\left(jL_{uv}\frac{\partial}{\partial v} - jL_{vv}\frac{\partial}{\partial u}\right)\left[j\frac{\partial(L_u, h)}{\partial(u, v)}\right]$$

$$= \left(jL_{uu}\frac{\partial}{\partial v} - jL_{uv}\frac{\partial}{\partial u}\right)\left[j\frac{\partial(L_v, h)}{\partial(u, v)}\right],$$

to eliminate $h(u, v)$ from (24) and (25) and obtain

$$\alpha_1\left[v\frac{\partial(L_v, j\{\partial(L_v, jW_1)/\partial(u, v) + \partial(L_u, jW_2)/\partial(u, v)\})}{\partial(u, v)}\right.$$

$$+u\frac{\partial(L_u, j\{\partial(L_v, jW_1)/\partial(u, v)+\partial(L_u, jW_2)/\partial(u, v)\})}{\partial(u, v)}\Bigg]$$

$$+\mu\left[\frac{\partial(L_v, jW_1)}{\partial(u, v)}+\frac{\partial(L_u, jW_2)}{\partial(u, v)}\right]-\rho(vW_1+uW_2)=0. \tag{28}$$

Collecting the results, we have:

THEOREM 1: *If $L(u, v)$ is the Legendre-transform function of a stream function of the equations of motion governing the plane steady flow of an incompressible fluid of the second grade, then $L(u, v)$ must satisfy*

$$\alpha_1\Bigg[v\frac{\partial(L_v, j\{\partial(L_v, jW_1)/\partial(u, v)+\partial(L_u, jW_2)/\partial(u, v)\})}{\partial(u, v)}$$

$$+u\frac{\partial(L_u, j\{\partial(L_v, jW_1)/\partial(u, v)+\partial(L_u, jW_2)/\partial(u, v)\})}{\partial(u, v)}\Bigg]$$

$$+\mu\left[\frac{\partial(L_v, jW_1)}{\partial(u, v)}+\frac{\partial(L_u, jW_2)}{\partial(u, v)}\right]-\rho[vW_1+uW_2]=0, \tag{29}$$

where W_1, W_2, j and ω are given by (26), (27) *and* (23). *Given a solution $L=L(u, v)$ of* (29), *we can find the velocity components as functions of (x, y) from* (21). *Vorticity, generalized energy function and pressure are then obtained from* (10) *and* (11).

It is also of some interest to develop the flow equations in polar coordinates (q, θ) in the hodograph plane. On writing

$$u+\mathrm{i}v=q\,\mathrm{e}^{\mathrm{i}\theta}, \tag{30}$$

we note the following transformations:

$$\frac{\partial}{\partial u}=\cos\theta\frac{\partial}{\partial q}-\frac{\sin\theta}{q}\frac{\partial}{\partial\theta},\qquad \frac{\partial}{\partial v}=\sin\theta\frac{\partial}{\partial q}+\frac{\cos\theta}{q}\frac{\partial}{\partial\theta}$$

$$\frac{\partial(F, G)}{\partial(u, v)}=\frac{\partial(F^*, G^*)}{\partial(q, \theta)}\cdot\frac{\partial(q, \theta)}{\partial(u, v)}=\frac{1}{q}\frac{\partial(F^*, G^*)}{\partial(q, \theta)}, \tag{31}$$

where $F(u, v)=F^*(q, \theta)$, $G(u, v)=G^*(q, \theta)$ are continuously differentiable functions. On using these relations, and regarding (q, θ) as new independent variables, the expressions for j, ω, W_1 and W_2, in the (q, θ) plane, become

$$j^*=q^4\left[q^2L^*_{qq}(qL^*_q+L^*_{\theta\theta})-(L^*_\theta-qL^*_{q\theta})^2\right]^{-1}, \tag{32}$$

$$\omega^*=j^*\left[L^*_{qq}+\frac{1}{q^2}L^*_{\theta\theta}+\frac{1}{q}L^*_q\right], \tag{33}$$

$$W_1^* = W_1(q\cos\theta, q\sin\theta) = \frac{1}{q}\,\frac{\partial\left(\sin\theta L_q^* + \dfrac{\cos\theta}{q}L_\theta^*, \omega^*\right)}{\partial(q,\theta)}, \tag{34}$$

$$W_2^* = W_2(q\cos\theta, q\sin\theta) = \frac{1}{q}\,\frac{\partial\left(\cos\theta L_q^* - \dfrac{\sin\theta}{q}L_\theta^*, \omega^*\right)}{\partial(q,\theta)}. \tag{35}$$

The terms involving the dynamic viscosity μ and normal-stress modulus α_1 are similarly transformed in the (q, θ) plane as

$$\frac{1}{q}\left\{\frac{\partial\left(\sin\theta L_q^* + \dfrac{\cos\theta}{q}L_\theta^*, j^*W_1^*\right)}{\partial(q,\theta)} + \frac{\partial\left(\cos\theta L_q^* - \dfrac{\sin\theta}{q}L_\theta^*, j^*W_2^*\right)}{\partial(q,\theta)}\right\}$$

$$= X^*(q,\theta) \quad \text{(say)} \tag{36}$$

and

$$q\sin\theta\left[\frac{1}{q}\,\frac{\partial\left(\sin\theta L_q^* + \dfrac{\cos\theta}{q}L_\theta^*, j^*X^*\right)}{\partial(q,\theta)}\right]$$

$$+q\cos\theta\left[\frac{1}{q}\,\frac{\partial\left(\cos\theta L_q^* - \dfrac{\sin\theta}{q}L_\theta^*, j^*X^*\right)}{\partial(q,\theta)}\right]. \tag{37}$$

Summarizing the results, we have:

COROLLARY: *If $L^*(q, \theta)$ is the Legendre transform function of a stream function of the equations of motion for the plane steady incompressible flow of a second-grade fluid, then $L^*(q, \theta)$ must satisfy*

$$\alpha_1\left[\sin\theta\frac{\partial\left(\sin\theta L_q^* + \dfrac{\cos\theta}{q}L_\theta^*, j^*X^*\right)}{\partial(q,\theta)}\right.$$

$$\left.+\cos\theta\frac{\partial\left(\cos\theta L_q^* - \left(\dfrac{\sin\theta}{q}\right)L_\theta^*, j^*X^*\right)}{\partial(q,\theta)}\right]$$

$$+\mu X^* - \rho q(\sin\theta W_1^* + \cos\theta W_2^*) = 0, \tag{38}$$

where j^*, ω^*, W_1^*, W_2^* and X^* are respectively given by (32), (33), (34), (35) *and* (36).

Given a solution $L^* = L^*(q, \theta)$ *of* (38), *we can determine* u, v *by making use of* (30), *and* (x, y) *are expressible as*

$$x = \sin\theta L_q^* + \frac{\cos\theta}{q} L_\theta^*, \qquad y = \frac{\sin\theta}{q} L_\theta^* - \cos\theta L_q^*. \tag{38a}$$

4. Illustrations

In this section we consider some of the applications of Theorem 1 and its corollary.

(1). As a first application we let

$$L(u, v) = Au^m + Bv^n \tag{39}$$

be the Legendre transform function where $m \neq 0$, $n \neq 0$, $m \neq 1$, $n \neq 1$ and where A, B are nonzero constants and m, $n \in \mathbb{R}$. If we substitute (39) in (23), (26) and (27), we obtain

$$\begin{aligned} j &= \left[nm(m-1)(n-1)ABu^{m-2}v^{n-2}\right]^{-1}, \\ \omega &= \frac{1}{Bn(n-1)v^{n-2}} + \frac{1}{Am(m-1)u^{m-2}}, \\ W_1 &= \frac{Bn(n-1)(m-2)u^{-m+1}v^{n-2}}{Am(m-1)}, \\ W_2 &= \frac{Am(m-1)(2-n)u^{m-2}v^{-n+1}}{Bn(n-1)}. \end{aligned} \tag{40}$$

On employing (39) and (40) in (29) we find that $L(u, v) = Au^m + Bv^n$ can be the Legendre transform of a stream function for a plane steady flow of a second-grade fluid, provided that (for all u and v) m and n satisfy

$$\begin{aligned} &\alpha_1\left[\frac{Am(m-1)(2-n)(3-2n)(4-3n)}{B^3n^3(n-1)^3} u^{m-1}v^{3-3n}\right. \\ &\left. + \frac{Bn(n-1)(m-2)(2m-3)(3m-4)}{A^3m^3(m-1)^3} u^{3-3m}v^{n-1}\right] \\ &+ \mu\left[\frac{B(n-1)n(m-2)(2m-3)}{A^2m^2(m-1)^2} u^{2-2m}v^{n-2}\right. \\ &\left. + \frac{Am(m-1)(2-n)(3-2n)}{B^2n^2(n-1)^2} u^{m-2}v^{2-n}\right] \\ &+ \rho\left[\frac{Bn(n-1)(2-m)}{Am(m-1)} u^{1-m}v^{n-1} + \frac{Am(m-1)(n-2)}{Bn(n-1)} u^{m-1}v^{1-n}\right] = 0. \end{aligned} \tag{41}$$

Equation (41) is satisfied only for $m = n = 2$, and (39) and (40) then become

$$L(u, v) = Au^2 + Bv^2,$$
$$j = \frac{1}{4AB}, \qquad \omega = \frac{A+B}{2AB}, \qquad W_1(u, v) = W_2(u, v) = 0. \tag{42}$$

Substituting (42) in (21), we find

$$u(x, y) = -y/2A, \qquad v(x, y) = x/2B, \tag{43}$$

and the stream lines and the pressure turn out to be, respectively,

$$x^2/(4B) + y^2/(4A) = \text{constant},$$

$$p(x, y) = \frac{\rho}{8AB}(x^2 + y^2) + \left(\frac{3\alpha_1 + 2\alpha_2}{8}\right)\frac{(A-B)^2}{A^2B^2} + p_0. \tag{44}$$

We remark that the stream lines are similar to those obtained for the viscous fluid but the pressure function is different from the viscous-fluid case.

(2). In the next example we consider

$$L(u, v) = u^m v^n, \tag{45}$$

to be the Legendre-transform function with $m \neq 0$, $n \neq 0$ and $m + n \neq 1$.

As before, using (48) in (23), (26) and (27), we find

$$\begin{aligned}
j &= [u^{2-2m}v^{2-2n}]/[mn(1-m-n)],\\
\omega &= \left\{\frac{(m-1)}{n(1-m-n)}v^2 + \frac{(n-1)}{m(1-m-n)}u^2\right\}u^{-m}v^{-n},\\
W_1 &= \frac{m(m-1)}{(1-m-n)}u^{-1} - \frac{n(n-1)(2n+m-2)}{m(1-m-n)}uv^{-2},\\
W_2 &= \frac{m(m-1)(2m+n-2)}{n(1-m-n)}vu^{-2} - \frac{n(n-1)}{(1-m-n)}v^{-1}.
\end{aligned} \tag{46}$$

On the substituting the above expressions in (29) we note that (45) can be the Legendre transform of a stream function for a plane steady flow of second-grade fluid provided that (for all u and v) m and n satisfy

$$4\alpha_1\left[\frac{(m-1)(1-n)(n-m)}{mn(1-m-n)^2}u^{1-2m}v^{1-2n}\right.$$
$$+\frac{n(n-1)(2n+m-2)(3n+2m-3)}{m^3(1-m-n)^2}u^{3-2m}v^{-2n}$$

$$-\frac{m(m-1)(2m+n-2)(3m+2n-3)}{n^3(1-m-n)^2}u^{-2m-1}v^{3-2n}\Bigg]$$

$$+\mu\left[\frac{2(m-1)(1-n)}{(1-m-n)^2}u^{-m}v^{-n}+\frac{n(n-1)(2n+m-2)(3n+2m-3)}{m^2(1-m-n)^2}u^{2-m}v^{-n-2}\right.$$

$$\left.+\frac{m(m-1)(2m+n-2)(3m+2n-3)}{n^2(1-m-n)^2}u^{-2-m}v^{2-n}\right]$$

$$+\rho\left[\frac{2m(m-1)}{n}u^{-1}v+\frac{2n(1-n)}{m}uv^{-1}\right]=0. \tag{47}$$

Clearly, (47) is satisfied if $m=n=1$, and (45), (46) then reduce to

$$\begin{aligned} &L(u, v)=uv, \\ &j=-1, \qquad \omega=0, \qquad W_1=W_2=0. \end{aligned} \tag{48}$$

where j, ω, W_1, W_2 are functions of (u, v).

On proceeding as in the previous example, we now find

$$u(x, y)=x, \qquad v(x, y)=-y \tag{49}$$

and

$$p(x, y)=-\tfrac{1}{2}\rho(x^2+y^2)+2(3\alpha_1+2\alpha_2)+C, \tag{50}$$

where C is an arbitrary constant. The stream lines are given by

$$yx=C_2, \tag{51}$$

which are rectangular hyperbolae.

We point out that the presence of the normal stress modulus α_1, that is the consideration of non-Newtonian nature of the fluid, eliminates two other possible solutions, namely $m=1$, $n=-1$, $6\mu=\rho$ and $m=-1$, $n=1$, $6\mu=\rho$ which are possible for viscous fluids [2].

(3). In the remainder of this section, we investigate the solutions of flow problems in (q, θ) coordinates. Let

$$L^*(q, \theta)=F(q) \tag{52}$$

be the Legendre-transform function such that $F'(q)\neq 0$, $F''(q)\neq 0$. On substituting (52) in (32), (33), (34), 35), we get

$$
\begin{aligned}
j^* &= \frac{q}{F'(q)F''(q)} = j^*(q),\\
\omega^* &= \frac{qF''(q)+F'(q)}{F'(q)F''(q)} = \omega^*(q),\\
W_1^* &= -\frac{F'(q)\cos\theta}{q}\left(\frac{qF''(q)+F'(q)}{F'(q)F''(q)}\right)' = \frac{F'\cos\theta}{q}\omega^{*\prime}(q),\\
W_2^* &= \frac{F'(q)\sin\theta}{q}\left(\frac{qF''(q)+F'(q)}{F'(q)F''(q)}\right)' = \frac{F'\sin\theta}{q}\omega^{*\prime}(q).
\end{aligned}
\tag{53}
$$

When these relations are employed in (36), (37) and (38), we find that the terms involving α_1, the normal-stress modulus, and the terms involving ρ, the density, both become identically zero, and we obtain the condition

$$
\omega^{*\prime} + F'\left(\frac{\omega^{*\prime}}{F''}\right)' = 0. \tag{54}
$$

For $\omega^*(q) \neq 0$, the above equation, after integrating twice with respect to q, yields

$$
\omega^*(q) = C \ln F' + D, \tag{55}
$$

where C and D are arbitrary constants. Using (53) and (38a) we find that

$$
q = k_1 r \ln r + k_2 r + k_3/r, \tag{56}
$$

where $r = \sqrt{x^2+y^2}$ and $k_1 = \frac{1}{2}C$, $k_2 = \frac{1}{4}(2D - C)$ and k_3 are arbitrary constants. With the help of (56) and (38a) we find that

$$
\begin{aligned}
u(x, y) &= -y\left[\tfrac{1}{2}k_1 \ln(x^2+y^2) + k_2 + \frac{k_3}{x^2+y^2}\right],\\
v(x, y) &= x\left[\tfrac{1}{2}k_1 \ln(x^2+y^2) + k_2 + \frac{k_3}{x^2+y^2}\right],\\
\omega(x, y) &= k_1 \ln(x^2+y^2) + k_1 + 2k_2.
\end{aligned}
\tag{57}
$$

Since $\omega(x, y)$ is harmonic, the terms involving α_1 in the linear momentum equations in system (11) are identically satisfied, and after considerable simplification, the pressure distribution turns out to be

$$
\begin{aligned}
p = \rho\Big\{&\left[\tfrac{1}{4}k_1k_3 + \tfrac{1}{8}k_1^2(x^2+y^2)\right]\left[\ln(x^2+y^2)\right]^2\\
&+ \left[\left(\tfrac{1}{2}k_1k_2 - \tfrac{1}{4}k_1^2\right)(x^2+y^2) + k_2k_3\right] \ln(x^2+y^2)
\end{aligned}
$$

$$+\left[\tfrac{1}{4}k_1^2+\tfrac{1}{2}k_2^2-\tfrac{1}{2}k_1k_2\right](x^2+y^2)-\frac{k_3^2}{2(x^2+y^2)}-k_2k_3\Big\}$$

$$-2\mu k_1\tan^{-1}\left(\frac{x}{y}\right)+2\alpha_1k_1\left[\tfrac{1}{2}k_1\ln(x^2+y^2)+k_2+\frac{k_3}{x^2+y^2}\right]$$

$$+\frac{3\alpha_1+2\alpha_2}{2}\left[k_1^2+\frac{4k_3^2}{(x^2+y^2)^2}-\frac{4k_1k_3}{x^2+y^2}\right]+p_0. \quad (58)$$

The stream function $\psi(x, y)$ is given by

$$\psi(x, y)=(x^2+y^2)\left[\tfrac{1}{4}k_1\ln(x^2+y^2)+\tfrac{1}{2}k_2-\tfrac{1}{4}k_1\right]+\tfrac{1}{2}k_3\ln(x^2+y^2)+C.$$

We point out that the pressure (58) is considerably different from the viscous-fluid case.

We note that (54) is also satisfied for $\omega^*=\text{constant}=\omega_0$. In this case the velocity components and vorticity are given as

$$u(x, y)=\frac{ky}{x^2+y^2}-\tfrac{1}{2}\omega_0 y,$$

$$v(x, y)=\frac{kx}{x^2+y^2}+\tfrac{1}{2}\omega_0 x, \quad (59)$$

$$-\nabla^2\psi=\omega_0,$$

and the stream function is given by

$$\psi(x, y)=f(x, y)-\tfrac{1}{4}\omega_0(x^2+y^2),$$

where $f(x, y)$ is a harmonic function.

(4). Now we investigate the solution of a flow problem when $L^*(q, \theta)$ is a function of θ only. We assume

$$L^*(q, \theta)=G(\theta) \quad (60)$$

to be the Legendre transform for the system of equations (38) such that $G'(\theta)\neq 0$. On employing (60) in (32), (33), (34) and (35) we find that

$$j^*(q, \theta)=\frac{-q^4}{G'^2(\theta)}, \qquad \omega^*(q, \theta)=-q^2\frac{G''(\theta)}{G'^2(\theta)},$$

$$W_1^*(q, \theta)=\frac{G'''(\theta)\cos\theta-2G''(\theta)\sin\theta}{qG'(\theta)}, \quad (61)$$

$$W_2^*(q, \theta)=-\frac{G'''(\theta)\sin\theta+2G''(\theta)\cos\theta}{qG'(\theta)}.$$

Using these relations in (38), we obtain the condition

$$4\alpha_1 q^2[G^{(4)}+4G'']=\mu G'^3[G^{(4)}+4G'']+2\rho G'^4G''. \tag{62}$$

The above equation is satisfied if

$$G^{(4)}+4G''=0 \quad \text{and} \quad \mu(G^{(4)}+4G'')+2\rho G'G''=0. \tag{63}$$

Thus

$$G(\theta)=A\theta+B, \tag{64}$$

where A, B are arbitrary constants, is the solution of (62).

Proceeding as before, we find

$$u(x, y)=\frac{Ax}{x^2+y^2}, \qquad v(x, y)=\frac{Ay}{x^2+y^2}. \tag{65}$$

The stream function $\psi(x, y)$ and pressure $p(x, y)$ take the form, respectively,

$$\psi(x, y)=\tan^{-1}\left(\frac{x}{y}\right)+C,$$

$$p(x, y)=C-\frac{\rho A^2}{2(x^2+y^2)}+\frac{3\alpha_1+2\alpha_2}{4}\left[8\left\{\frac{\partial}{\partial x}\left(\frac{Ax}{x^2+y^2}\right)\right\}^2+\frac{32A^2x^2y^2}{(x^2+y^2)^4}\right]. \tag{66}$$

(5). Finally we consider the case when

$$L^*(q, \theta)=q^2G(\theta). \tag{67}$$

Following the previous examples, we note that

$$j^*=[4G^2+2GG''-G'^2]^{-1}=j^*(\theta),$$

$$\omega^*=\frac{4G+G''}{[4G^2+2GG''-G'^2]}=\omega^*(\theta),$$

$$W_1^*=\frac{\omega^{*\prime}}{q}(2G\sin\theta+G'\cos\theta),$$

$$W_2^*=\frac{\omega^{*\prime}}{q}(2G\cos\theta-G'\sin\theta).$$

Substituting these relations in (38), we get

$$2\alpha_1[Gj^*\{(4G^2+G'^2)j^*\omega^{*\prime}\}']'$$

$$\mu[(4G^2+G'^2)j^*\omega^{*\prime}]'-2\rho G\omega^{*\prime}q^2=0. \tag{68}$$

Equation (68) is satisfied if

$$2\alpha_1\left[Gj^*\{(4G^2+G'^2)j^*\omega^{*\prime}\}'\right]'+\mu\left[(4G^2+G'^2)j^*\omega^{*\prime}\right]'=0 \tag{69}$$

and

$$2\rho G\omega^{*\prime}=0. \tag{70}$$

Since $G \neq 0$, $\rho \neq 0$; therefore, $\omega^* = \text{constant} = \omega_0$. Hence, the solution and the rest of the analysis is the same as in the viscous-fluid case [2], except that the pressure is now given by

$$p(x, y)=\frac{-\rho}{2k_3}(x^2+y^2)+8(3\alpha_1+2\alpha_2)\frac{k_1^2+k_2^2}{k_3^2}+p_0. \tag{71}$$

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